Representation Theory, Part II: Character Theory

Jay Havaldar

Note that in S_3 the eigenvalues of the group elements completely determined the representation. If the eigenvalue was -1, it was the alternating representation; 1, the trivial representation; and finally ω, ω^2 correspond to the standard representation.

Note that if you know the eigenvalues λ_i of g, then you know the eigenvalues of $g^k = \lambda_i^k$ since the matrix is diagonal in some basis. Similarly, suppose that we know the sums $\sum \lambda_i^k$ for each k. Then this tells us everything about the eigenvalues.

For example, note that:

$$\left(\sum \lambda_i\right)^2 = \sum \lambda_i^2 + 2\sum_{i\leq j} \lambda_i \lambda_j$$

Now, suppose that the representation has dimension 2. Then the latter term on the right hand side is the middle term in the characteristic polynomial for g. The final term is the determinant of g. So with this information, we can solve for the eigenvalues of g.

Definition: The character of a representation is:

$$\chi_V(g) = \text{tr}(g|_V)$$

This is a class function, i.e. χ_V is constant within a conjugacy class, since similar matrices have the same trace.

2.1 Properties of Characters

The following give us a useful way to compute combinations of representations.

$$\begin{aligned} \chi_{V\oplus W} &= \chi_V + \chi_W \\ \chi_{V\otimes W} &= \chi_V \cdot \chi_W \\ \chi_{V^*} &= \chi_V(g^{-1}) = \overline{\chi_V(g)} \end{aligned}$$

The last property follows from the fact that the eigenvalues of g are all roots of unity.

Proposition 1 (The Original Fixed Point Formula)

If V is the permutation representation of G acting on a finite set X, then $\chi_V(g)$ is the number of elements fixed by g.

To prove this, consider that each orbit is a subspace of V which is fixed under G, hence a sub-representation. And with respect to the natural basis, the matrix has trace 0, unless the matrix is simply 1×1, in which case it has trace 1. Therefore, the trace of g is the number of fixed points.

2.2 Character Tables

A character table has conjugacy classes (with sizes above them) labeling the columns. The rows are labelled by irreducible representations. In the appropriate box is the character of a conjugacy class representative in the given representation.

Character Table for S_3 So far, the table looks like:

	1	3	2
S_3	1	(12)	(123)
U (trivial)	1	1	1
U' (alt.)	1	-1	1

To find the remaining row, we first note that the permutation representation is given by $(3, 1, 0)$ by the fixed point formula. However, the permutation representation is the direct sum of $U \oplus V$, where V is the standard representation. So using this information, we solve for V to obtain:

	1	3	2
S_3	1	(12)	(123)
U (trivial)	1	1	1
U' (alt.)	1	-1	1
V (std.)	2	0	-1

2.3 The First Projection Formula

Definition: The elements of a representation V which are fixed by each $g \in G$ is denoted V^G.

Proposition 2

The following map projections V onto V^G:

$$\varphi(v) = \frac{1}{|G|} \sum_{g \in G} gv$$

Furthermore, this map is G-linear and projects onto V^G. In particular, each element of V^G spans a one-dimensional trivial representation. So with a given representation, we can find the multiplicity m of the trivial representation by noting that:

$$m = \dim V^G = \text{tr}(\varphi) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)$$

In particular, if V is irreducible and non-trivial, then this sum is zero.

Note also that:

$$Hom(V, W)^G = \{G\text{-module homomorphisms from } V \text{ to } W\}$$

So if V is irreducible, then by Schur's Lemma, each element of $Hom(V, W)$ is either a direct sum of isomorphisms or else 0. So in particular, $\dim Hom(V, W)^G$ is the multiplicity of V in W.

If both V, W are irreducible, then:

$$\dim Hom_G(V, W) = \begin{cases} 1 & V \cong W \\ 0 & V \ncong W \end{cases}$$

Now, note that $Hom(V, W)) = V^* \otimes W$, so that:

$$\chi_{Hom(V,W)}(g) = \overline{\chi_V(g)} \cdot \chi_W(g)$$

Now now, note that by Proposition 2, we have:

$$\dim Hom(V,W)^G = \frac{1}{|G|}\sum \chi_{Hom(V,W)}(g) = \frac{1}{|G|}\sum = \overline{\chi_V(g)} \cdot \chi_W(g)$$

This naturally lets us define the following inner product on class functions on G.

$$(\alpha, \beta) = \frac{1}{|G|}\sum \overline{\alpha(g)}\beta(g)$$

And according to this basis, we have just proved the following theorem.

Theorem (Schur Orthogonality 1)

The characters of irreducible representations of G are orthonormal.

Corollary

The number of irreducible representations is at most the number of conjugacy classes.

To see this, note that the dimension of the space of class functions is at most the number of conjugacy classes, and that irreducible representations correspond to irreducible characters, which are orthonormal.

Note also that if we have as a decomposition into irreducible representations:

$$V = a_1V_1 \oplus \cdots \oplus a_nV_n$$

Then the character is given by:

$$\chi_V = a_1\chi(V_1) + \cdots + a_n\chi(V_n)$$

Proposition 3 (Projection Formula) And in particular, since irreducible characters are orthonormal the inner product is given by:

$$(\chi_V, \chi_V) = \sum a_i^2$$

And finally, $(\chi_V, \chi_V) = 1$ iff V is irreducible. Similarly, we can write:

$$a_i = (\chi_V, \chi_{V_i})$$

2.4 Character of the Regular Representation

Let the regular representation be R. Really G is acting on itself by left multiplication, so the fixed point formula gives us:

$$\chi_R(g) = \begin{cases} 0 & g \neq e \\ |G| & g = e \end{cases}$$

So R is not irreducible so long as $G \neq \{e\}$. In particular, to get the ith coefficient of the regular representation, we can write:

$$a_i = (\chi_{V_i}, \chi_R) = \frac{1}{|G|}\chi_{V_i}(e)\chi_R(e) = \dim V_i$$

Corollary

> Each irreducible representation V_i of G appears in the regular representation $\dim V_i$ times.

Note however that:

$$\chi_R(e) = |G| = \sum a_i \dim(V_i) = \sum \dim(V_i)^2$$

And taking the character of any other element g

$$0 = \sum a_i \chi_{V_i}(g) = \sum \dim V_i \cdot \chi_{V_i}(g)$$

Theorem (Schur Orthogonality 2)

> The columns of a character table are also orthogonal.

This follows from matrix multiplication. In particular, for $g \in G$:

$$\sum_{\chi} \overline{\chi(g)}\chi(g) = \frac{|G|}{c(g)}$$

Where $c(g)$ is the size of the conjugacy class of g.

2.5 Abelian Subgroups and Products of Groups

Proposition 4

> Let $A \leq G$ be an abelian subgroup. Then each irreducible representation of G has degree at most $[G : A]$.

Suppose that we have an irreducible representation V of G. Then we can restrict the representation to A to obtain a representation of A.

Let $W \subseteq V$ be an irreducible subrepresentation of A. Since A is abelian, $\dim W = 1$ and it is spanned by some vector w. So we define:

$$V' = \{gw \ : \ g \in G\}$$

More precisely, we take the span of the orbit of w under G. Evidently, V' is stable under the action of G, and therefore is a subrepresentation of V. But V is irreducible, so $V' = V$. In particular we have for any $g \in G$, and $a \in A$:

$$ga(w) = g(\lambda w) = \lambda gw$$

For some $\lambda \in \mathbb{C}$. So the number of linearly independent elements gw is at most $[G : A]$, and the span of the orbit of w is exactly $V' = V$.

Proposition 5

> Suppose we have a product of groups $G_1 \times G_2$. Then if we have two irreducible representations V, W of G_1, G_2 respectively, then $V \otimes W$ is an irreducible representation of $G_1 \times G_2$.

We can prove this by simply noting that:

$$\frac{1}{|G_1|} \sum |\chi_V(a)|^2 = \frac{1}{|G_2|} \sum |\chi_W(b)|^2 = 1$$

And since the character of a tensor product is the product of characters we get:

$$\frac{1}{|G_1||G_2|} \sum \sum |\chi_V(a)|^2 |\chi_W(b)|^2 = 1 \cdot 1 = 1$$

And so indeed the tensor product representation is irreducible.

Proposition 6

> Each irreducible representation of $G_1 \times G_2$ is of the form $V \otimes W$ for some V irreducible representation of G_1 and W irreducible representation of G_2.

Let f be a class function on $G_1 \times G_2$ which is orthogonal to each $V \otimes W$, where V, W irreducible in G_1, G_2 respectively. Note that since multiplication is defined componentwise we have:

$$(g, h)(a, b)(g^{-1}, h^-1) = (gag^{-1}, hah^{-1})$$

We consider the inner product of f with χ_V and χ_W. We have:

$$\sum_{a,b} f(a,b)\chi_V(a)^*\chi_W(b)^* = 0$$

Now fix:

$$g(a) = \sum_b f(a,b)\chi_V(a)^*$$

Let $h \in G_1$. Then we have:

$$g(hah^{-1}) = \sum_b f(hah^{-1},b)\chi_W(b)^*$$

But:

$$(hah^{-1},b) = (h,1)(a,b)(h,1)^{-1}$$

And f is a class function, so indeed we get:

$$g(hah^{-1}) = g(a)$$

So g is a class function on G_1. However we have for irreducible character χ_V on G_1:

$$\sum_a g(a)\chi_V(a)^*$$

And so $g = 0$ identically. We can argue similarly to complete the proof and say that $f(a,b)$ is identically zero.