Analysis, Part III: Uniform Functions on Metric Spaces

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We know that the following holds for sequences in a metric space, if f is continuous:

$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$$

The weakest form of convergence is **pointwise convergence**.

Definition: $f_n \to f$ pointwise if for every x and for every $\epsilon > 0$, there exists N so that if $n \geq N$ then $d(f_n(x), f(x)) < \epsilon$.

But this definition does not allow us to swap the order of limits.

A stronger condition, however, does allow this:

Definition: $f_n \to f$ pointwise if for every $\epsilon > 0$, there exists N so that for all x, if $n \geq N$ then $d(f_n(x), f(x)) < \epsilon$.

The key distinction here is that the value of N works for all x. As alluded to earlier, we have the following proposition.

Proposition

A sequence of continuous functions converges uniformly to a continuous function.

Corollary

$$\lim_{n\to\infty} \lim_{x\to x_0} f_n(x) = \lim_{x\to x_0} \lim_{n\to\infty} f(x)$$

We also have this nice theorem:

Proposition

Let f_n be a sequence of continuous functions which converge uniformly to f. Then for any sequence x_n which converges to x, $f_n(x_n)$ converges to $f(x)$.

Similarly, uniform limits preserve boundedness in a nice way.

Proposition

A sequence of bounded functions converges uniformly to a bounded function.

3.1 Metrics of Uniform Convergence

Definition: Let X, Y be metric spaces. Let $B(X \to Y)$ denote the space of bounded functions from $X \to Y$. We let:

$$d_\infty(f, g) = \sup_{x\in X} d(f(x), g(x))$$

This is called the L^∞ metric or the ``sup norm'' metric. Since f, g are assumed bounded in this space, $d(f, g)$ is bounded.

This space with this norm has the nice property that convergence in this space directly correlates with uniform convergence in the regular sense.

Proposition

> Let f_n be a sequence of functions in $B(X \to Y)$; then f_n converges to $f \in B(X \to Y)$ if and only if f_n converges to f uniformly.

In particular, we define the continuous and bounded functions.

Definition: $C(X \to Y)$ is the space of bounded and continuous functions from X to Y. This a closed subspace of $B(X \to Y)$

And finally, we have Cauchy sequences:

Proposition

> If Y is a complete metric space, then the space $C(X \to Y)$ is a complete subspace of $B(X \to Y)$.

3.2 The Weierstrass M-Test

Definition: A series of functions f_n converges pointwise to $f(x)$ if the partial sums converge pointwise to $f(x)$

In a similar way we have uniform convergence:

Definition: A series of functions f_n converges uniformly to $f(x)$ if the partial sums converge uniformly to $f(x)$

We can find an easy condition for uniform convergence using the following norm:

Definition: If $f : X \to \mathbb{R}$ is a bounded function, define the sup norm to be the number:

$$|f|_\infty = \sup_{x \in X} f(x)$$

And finally we have the immensely useful Weierstrass M-Test:

Theorem

> Let f_n be a series of bounded real-valued functions such that $\sum_{n=1}^{\infty} |f_n|_\infty$ is convergent.
>
> Then the series $\sum_{n=1}^{\infty} f_n$ converges uniformly to a continuous function f on X.

3.3 Uniform Convergence & Integration/Differentation

Theorem

> Let $[a, b]$ be an interval and f_n Riemann integrable functions. Suppose f_n converges uniformly on $[a, b]$ to a function f. Then f is also Riemann integrable and:
>
> $$\lim_{n\to\infty} \int_{[a,b]} f_n = \int_{[a,b]} \lim_{n\to\infty} f_n = \int_{[a,b]} f$$

An analogy of this theorem exists for series.

Corollary

> Let $[a, b]$ be a an interval. Let f_n be a sequence of Riemmann integrable functions on $[a, b]$ such that the series $\sum\limits_{n=1}^{\infty} f_n$ is uniformly convergent. Then we can say:
>
> $$\sum_{n=1}^{\infty} \int_{[a,b]} f_n = \int_{[a,b]} \sum_{n=1}^{\infty} f_n$$

3.3.1 Derivatives and Uniform Convergence

We ask if the same is true for derivatives. If f_n converges uniformly to f, then if we require f_n to be differentiable is f differentiable? And do f_n' converge to f'? In general, the answer is no.

As a counterexample consider:

$$f_n(x) = n^{1/2} \cos nx$$

This sequences converges uniformly to 0 if we look at the sup norm. However, its derivative at the origin is never 0. Thus, f_n' need not converge to f'.

Similarly, consider the sequence:

$$f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$$

We can check that this sequence converges uniformly to $|x|$, which is not differentiable at the origin. This, however, is the theorem we're looking for -- it gives us more or less the converse, so long as we have pointwise convergence at least at one point.

Theorem

> Let $[a, b]$ be an interval and f_n be differentiable functions with continuous derivative on the interval. Suppose the derivatives f_n' converge uniformly to a function $g : [a, b] \to \mathbb{R}$.

Suppose also that there exists a point x_0 so that the limit $\lim_{n\to\infty} f_n(x_0)$ exists (i.e. we have pointwise convergence at some point). Then the functions f_n converge uniformly to a differentiable function f such that the derivative $f' = g$.

Corollary

Let $[a, b]$ be an interval and f_n be differentiable functions with continuous derivative on the interval. Suppose the series $\sum_{n=1}^{\infty} |f_n'|$ converges absolutely.

Suppose also that there exists a point x_0 so that $\sum_{n=1}^{\infty} f_n(x_0)$ converges. Then the series $\sum_{n\to\infty} f_n$ converges uniformly on $[a, b]$ to a differentiable function f such that:

$$\frac{d}{dx} \sum_{n\to\infty} f_n(x) = \sum_{n\to\infty} \frac{d}{dx} f_n(x)$$

Example Let $f : \mathbb{R} \to \mathbb{R}$ be the function:

$$f(x) = \sum_{n=1}^{\infty} 4^{-n} \cos(32^n \pi x)$$

This series is uniformly convergent (Weierstrass M-Test) and continuous, but nowhere differentiable.