Analysis, Part II: Continuous Functions on Metric Spaces

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Definition: A function $f : X \to Y$ is continuous at x_0 if for every $\epsilon > 0$, there exists a $\delta > 0$ so that:

$$d(x, x_0) < \delta \implies d(f(x), f(x_0)) < \epsilon$$

Theorem

> The following are equivalent: - f is continuous at x_0. - If $x_n \to x_0$, then $f(x_n) \to f(x_0)$. - For every open set $V \subset Y$ containing $f(x_0)$, there is an open set $U \subset X$ containing x_0 so that $f(U) \subset V$.

We can generalize this statement for functions which are everywhere continuous:

Theorem

> The following are equivalent: - f is continuous. - If $x_n \to x_0$, then $f(x_n) \to f(x_0)$. - Whenever V is an open set in Y, $f^{-1}(V)$ is open in X. - Whenever F is a closed set in Y, $f^{-1}(F)$ is closed in X.

This tells us that the topology definition of open sets is exactly the same as the definition on metric spaces (used for calculus). Without a metric, we still have the definition about open sets.

As expected, compositions of continuous functions are continuous; addition, substraction, multiplication, max, and min are all continuous.

Also as expected, we can take direct sumss of functions. $f \oplus g$ is continuous at a point iff f, g are both continuous at x_0.

2.1 Continuity and Compactness

Continuous functions interact well with compact sets in the following way.

Theorem

> The image of a compact set under a continuous function is compact.

So continuity preserves compactness. The following property of compact sets is important.

Proposition

> Let f be a continuous function on a compact set. Then, f is bounded, and it attains its maximum (and minimum) somewhere within the compact set.

A more useful generalization of continuity is uniform continuity -- the key in this definition is that it does NOT depend on x.

Definition: f is uniformly continuous if for every $\epsilon > 0$ there exists a $\delta > 0$ so that:

$$d(x, y) < \delta \implies d(f(x), f(y)) < \epsilon$$

For any choice of x, y.

And on a compact domain these two notions are equivalent!

Theorem

Let X be compact. Then $f : X \to Y$ is continuous iff it is uniformly continuous.

2.2 Connectedness

Definition: X is disconnected if there exist disjoint non-empty open sets V, W so that $X = V \cup W$. X is connected if it is nonempty and not disconnected.

Note that in this definition, V, W are relatively open with respect to X. If there exists a larger space $X \subset Y$, To say that V is open relatively with respect to X is to say that $V = X \cap V'$ for some open set $V' \subset Y$. So this means that the definition of connectedness is *intrinsic* and does not depend on the ambient space.

Corollary

X is connected iff the only clopen sets are X itself and the empty set.

On the real line, the connected sets are precisely intervals.

Finally, continuity preserves connectedness just like it does compactness.

Theorem

Let $f : X \to Y$ be a continuous map. Let $E \subset X$ be connected. Then $f(E)$ is connected.

The simple reason why is from the definition; we can directly map back a disjoint union of open sets to a disjoint union of open sets.

As a direct corollary, we get the Intermediate Value Theorem.

Corollary

Let $f : X \to \mathbb{R}$ be continuous. Let E be connected in X, with $a, b \in E$. If $y \in (f(a), f(b))$, then there exists $c \in E$ so that $f(c) = y$.