Topology, Part 2: Continuous Functions

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Now that we have defined the basic structure of a topological space, we are ready to consider functions between spaces. We begin with an important condition: continuity.

Definition: A function $f : X \to Y$ between two topological spaces is continuous if, for each open set $V \subset Y$, $f^{-1}(V)$ is open in X.

Because of what we know about bases, it is sufficient to check that the inverse image of every basis element in Y is open.

Theorem

> The following are equivalent. - f is continuous. - For any subset $A \subset X$, $f(\bar{A}) \subset \overline{f(A)}$. - For a closed set $B \subset X$, $f^{-1}(B)$ is closed in X. - Take $x \in X$. Then for every neighborhood V of $f(x)$, there is a neighborhood U of x so that $f(U) \subset V$.

We won't prove all the parts of this theorem, as it basically reduces to set theory. Note however, that continuous functions preserve the topology of a space (both open and closed sets). Furthermore, the last bullet point here corresponds to our usual definition of continuity on the real line, if we define the standard topology on $\mathbb{R}$:

- f is continuous if for each point x_0 and each $\epsilon > 0$, there exists a $\delta > 0$ so that if x lies within a distance of δ from x_0, then $f(x)$ lies within a distance of ϵ fom $f(x_0)$.

Definition: Let $f : X \to Y$ be a bijection. Then, if f, f^{-1} are both continuous, f is called a **homeomorphism**.

Homeomorphisms, as we will see, are the most important types of maps in topology. A **topological property** is defined as one which is preserved by homeomorphisms. Homeomorphisms play the same role in topology as isomorphisms do in algebra.

Definition: Let $f : X \to Y$ be continuous and injective, and the restriction of the range to $f(X)$ is bijective. If f^{-1} is a homeomorphism, then the map is called an **imbedding** of X in Y.

The following theorem gives us sufficient conditions for continuity.

Theorem

> - A constant map is continuous.
> - If A is a subspace of X, then the inclusion map from A to X is continous.
> - The composition of two continuous functions is continuous.
> - If $f : X \to Y$ is continuous, then the restriction of f to a subspace A of X is continuous.
> - Similarly, restricting the range of a continuous function to a subspace is continuous.
> - $f : X \to Y$ is continuous if X is the union of open sets U_α so that the restriction of f to each U_α is continuous.

The last point tells us that, in particular, we can check continuity on basis elements.

Theorem (Pasting Lemma)

> Let $X \to A \cup B$, where A, B are closed in X. Then, suppose that $f : A \to Y$ and that $g : B \to Y$ are continuous, and that they agree on the intersection $A \cap B$. Then f and g combine to create a continuous function $h : X \to Y$, with:
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> $$h(x) \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

The proof is fairly simple. Take a closed subset of Y. Then $C = f^{-1}(C) \cup g^{-1}(C)$. Since f is continuous, we know that $f^{-1}(C)$ is closed in A. Since A is closed in X, we finally have that $f^{-1}(C)$ is closed in X. A similar argument goes for g; so the intersection of the two sets is indeed closed.

We could construct manifolds by using the theorem above, but instead creating X as a union of closed sets and repeating the argument verbatim.

Theorem

> Let $f : A \to X \times Y$. Then $f = (f_1(a), f_2(a))$ is continuous if and only if its coordinate functions f_1, f_2 are continuous.