Topology, Part I: Topological Spaces

Jay Havaldar

0.1 Definition of a Topology

Definition: Let X be a set. We define a **topology** $\mathcal{T}$ on X to be a collection of sets (called **open sets**) so that: - $\emptyset \in \mathcal{T}$ - $X \in \mathcal{T}$ - $\mathcal{T}$ is closed under unions and finite intersections.

We name some examples:

- If X is a nonempty set, we can define T to be every subset of X. This is called the **discrete topology** on X.

Definition: Let X be a set with two topologies $\mathcal{T}_1, \mathcal{T}_2$. If $\mathcal{T}_1 \subset \mathcal{T}_2$ or $\mathcal{T}_2 \subset \mathcal{T}_1$, we say that they are **comparable**. If $\mathcal{T}_1 \subset \mathcal{T}_2$ it is **finer** than $\mathcal{T}_2$; if $\mathcal{T}_1 \supset \mathcal{T}_2$ we say it is **coarser**.

Definition: Let X be a topological space and $x \in X$. An open set containing x is called a **neighborhood** of x. We now come to a familiar definition of open sets.

Theorem

> Let X be a topological space and $A \subset X$. Then A is open in X iff for every $x \in A$, there is a neighborhood U of x so that $U \subset A$.

We prove the theorem as follows. Suppose that A is open and $x \in A$. Then we have found a neighborhood of X contained in A; namely, A itself.

Now conversely, suppose that for every $x \in A$, there is a neighborhood $U_x \subset A$ so that $x \in U_x$. Then, $A = \bigcup\limits_{x \in A} U_x$; since each U_x is open, A is as well.

0.2 Bases

Definition: Let X be a topological space. We say that a collection of subsets $\mathcal{B}$ of X is a **basis** if:

- For every $x \in X$, $x \in B$ for some $B \in \mathcal{B}$; the collection $\mathcal{B}$ covers X.
- For every $x \in X$, if $x \in B_1 \cap B_2$ for $B_1, B_2 \in \mathcal{B}$, then there is a set $B_3 \in \mathcal{B}$ so that $x \in B_3 \subset B_1 \cap B_2$.

We get a topology out of a basis:

Definition: Let $\mathcal{B}$ be a basis for a set X. Then we define the **topology generated by** $\mathcal{B}$ as follows: the open sets consist of the empty set and every union of basis elements.

In particular, each basis element is an open set in the topology generated by the basis.

We still have to prove that the topology generated by $\mathcal{B}$ is indeed a topology. We begin with a lemma.

Lemma

> Let $\mathcal{B}$ be a basis. Then if $B_1, B_2, \ldots, B_n \in \mathcal{B}$, and $x \in \bigcap\limits_{i=1}^{n} B_i$, there exists a basis element B' so that $x \in B' \in \bigcap\limits_{i=1}^{n} B_i$

The $n = 2$ case follows from the definition of a basis. The other cases follow by induction.

Theorem

Let $\mathcal{B}$ be a basis for a set X. Then the topology generated by $\mathcal{B}$ is a topology.

We check the four properties of a topology. By definition, the empty set is indeed open. Furthermore, since every point $x \in X$ is contained in some basis element, X is the union of all the basis elements.

Since the open sets are unions of basis elements, unions of open sets are also unions of basis elements, and hence open.

We are left to show that finite intersections are open. Let $V = \bigcap_{i=1}^{n} U_i$, where each U_i is open and nonempty.

For each $x \in V$, $x \in U_i$ for all i. Since each U_i is a union of basis elements, in particular we can find a basis element B_i so that $x \in B_i \subset U_i$ for each U_i. Therefore, we can write $x \in \bigcap_{i=1}^{n} B_i \subset V$. By the definition of a basis and our earlier lemma, $x \in B_x$ for some $B_x \subset V$.

Therefore, $V = \bigcap_{x \in V} B_x$; since V is a union of basis elements, it is open as well.

We name some examples of bases and the topologies they generate:

- On the real line, the open intervals (a, b) form a basis. This basis generates the **standard topology** on $\mathbb{R}$, where the open sets are unions of intervals.
- Let X be a set. Then the singleton sets form a basis. This basis generates the discrete topology on X.
- On $\mathbb{R}^{\nvDash}$, we define a basis as open balls of radius $\epsilon > 0$. This basis generates the topology for which the open sets are unions of open balls; we can generate this same topology with open intervals of the form $(a, b) \times (c, d)$. This is called the **standard topology** on $\mathbb{R}^n$.

Theorem

Let X be a set and $\mathcal{B}$ a basis for a topology on X. Then U is open in the topology generated by $\mathcal{B}$ iff for each $x \in U$, there is a basis element B_x so that $x \in B_x \subset U$.

The forward direction follows from the fact that open sets are unions of basis elements. To prove the converse, we assume that for each $x \in U$, there is a basis element B_x so that $x \in B_x \subset U$. Then $U = \bigcup_{x \in U} B_x$, so indeed U is open.

We now give a way to construct a basis for a topology.

Theorem

> Let X be a set with a topology $\mathcal{T}$, and let $\mathcal{C}$ be a collection of open sets in X. Suppose that for each open set U and each point $x \in U$, there is an open set $V \in \mathcal{C}$ so that $x \in V \subset U$. Then C is a basis that generates the topology $\mathcal{T}$.

0.3 Special Kinds of Topologies

We discuss two kinds of standard topologies we can create from spaces.

0.3.1 The Product Topology

Definition: Suppose we have two topological spaces X, Y. Then we define the **product topology** on $X \times Y$ as the topology generated by the basis
$U \times V$, where U is open in X and V is open in Y.

We can define this topology equivalently with the following theorem.

Theorem

> If $\mathcal{B}, \mathcal{C}$ are bases for X, Y, respectively, then the collection:
>
> $$\mathcal{D} = \{B \times C | B \in \mathcal{B}, C \in \mathcal{C}\}$$
>
> Is a basis for the product topology on $X \times Y$.

0.3.2 The Subspace Topology

Definition: Let X be a topological space with topology $\mathcal{T}$. If $Y \subset X$, then the collection:

$$\mathcal{T}_Y = \{Y \cap U | U \in \mathcal{T}\}$$

Is a topology on Y, called the **subspace topology**.

Lemma

> If $\mathcal{B}$ is a basis for a topology on X, then we can construct a basis for the subspace topology on Y by taking:
>
> $$\mathcal{B}_Y = \{Y \cap B | B \in \mathcal{B}\}$$

Under special conditions, open sets in Y can also be open in X:

Lemma

> Let Y be a subspace of X. If U is open in Y, and Y is open in X, then U is open in X.

Since U is open in Y, $U = Y \cap V$ for some open set $V \in X$. But Y is open, so that means the intersection $Y \cap V$ is open as well.

We can connect the subspace topology and the product topology in the way you might expect:

Theorem

> Let $A \subset X, B \subset Y$. Suppose we take the product topology on $X \times Y$, and use this to construct the subspace topology on $A \times B$. Then this is exactly the product topology of $A \times B$.

In other words, this theorem tells us that the subspace of a product is indeed the product of subspaces, considered as topologies. Indeed the bases of both constructions are exactly the same.

0.4 Closed Sets, Interior, Closure, Boundary

Definition: A subset $A \subset X$ is called **closed** if its complement $X - A$ is open.

For example, $\emptyset, X$ are both closed and open, or clopen. Rectangles and circles in the plane with borders are also closed. We have some analogous properties to open sets:

Theorem Let X be a topological space. Then the closed sets satisfy: - $\emptyset$ is closed. - X is closed. - Intersections and finite unions of closed sets are closed.

The following property guarantees that every singleton set is closed, as is true on the plane with the standard topology.

Definition: A topological space X is called **Hausdorff** if for every pair fo distinct points $x, y \in X$, there are disjoint neighborhoods U, V of x and y respectively.

Examples include:

- The real line with the standard topology is Hausdorff.
- Any set with the discrete topology is Hausdorff.

Theorem

> If X is Hausdorff, then every singleton set is closed.

Pick an arbitrary element $y \in X - x$. Since X is Hausdorff, there are open sets U, V containing x, y respectively. So every $y \in X - x$ is contained in an open set; therefore $X - x$ is open.

Finally, we give a criteria for closed sets in the subspace topology.

Theorem

> Let Y be a subspace in X. Then a set A is closed in Y iff it is the intersection of Y and a closed set in X.

We now define the smallest closed set containing an open set, with the following set of definitions.

Definition: The interior of A is the union of all open sets containing A.

Evidently, if A is open, its interior is itself.

Definition: The closure of $\bar{A}$ is the intersection of all closed sets containing A.

So we have:

$$\text{Int}(A) \subset A \subset \bar{A}$$

We can define the following equivalent definitions, which are perhaps more useful.

Theorem

> Let A be a subspace of a topological space X, which has a basis $\mathcal{B}$. Then: - $x \in \bar{A}$ if every open set U in A intersects A. - $x \in \bar{A}$ iff every basis element containing x intersects A.

0.4.1 Limit Points

An alternate construction of closed sets comes from the concept of a limit point.

Definition: x is a **limit point** of A if every neighborhood of x intersects A in some point other than x itself.

Equivalently, using our prior theorem, $x \in \overline{A - x}$ iff it is a limit point.

Theorem

> A set in a topological space is closed iff it contains all its limit points.

Proof We consider A' to be the set of limit points of A. We already know that, by definition, $A \cup A' \subset \bar{A}$, since a neighborhood of a limit point intersects A. We want to show that indeed the opposite inclusion is true, meaning that $A \cup A' = \bar{A}$.

Conversely, we assume that $x \in \bar{A}$. Then, there are two cases. If $x \in A$, then we are done. If $x \notin A$, then by definition, any neighborhood x intersects A in some point other than x itself. Indeed, that means $x \in A'$.

Now we can prove our theorem. A set A is closed iff $\bar{A} = A$. But then, we have $A' \subset A$; so all the limit points of A are indeed in A.

So, a closed set guarantees the existence of limit points. The Hausdorff condition, in a way, guarantees the uniqueness of limit points.

Theorem

> If X is Hausdorff, then x is a limit point of a set A iff every neighborhood of x contains infinitely many points of A.

One direction is evident. We seek to prove that if x is a limit point, then any neighborhood contains infinitely many points. Suppose to the contrary that a neighborhood intersects A in finitely many points; then we can successively subtract these points from X and maintain open sets, since any finite set of points is closed. We are left with a neighborhood of x which intersects A in only x.

Corollary

> If X is Hausdorff, then a sequence of points converges to at most one point in X.

This follows from the above fact that a neighborhood of a limit point in a Hausdorff space indeed contains infinitely many points of the sequence. Suppose that we have two distinct limit points. Then they have two distinct neighborhoods. One neighborhood contains all points of the sequence except finitely many; so the other neighborhood cannot have infinitely many points.

Definition: A boundary point is in both $\bar{A}$ and $\overline{X - A}$.