Differential Equations, Part III: Higher Order Linear Equations

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We move onto higher order linear differential equations. As before, we need some suitable existence and uniqueness theorem which works for the general case.

0.1 Existence and Uniqueness

In general, a linear differential equation is of the form:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_0 y = g(t)$$

Where a_n are functions and $y^{(n)}$ represents the nth derivative of y.

Theorem

Suppose we have a linear differential equation with coefficients $a_i(t)$ and right hand side equal to $g(t)$ defined in an interval I. Then, given n initial conditions $y^{(n)}(t_0) = c_n$, there exists a unique solution to the differential equation in the interval.

Note that we already know how to solve this equation in many cases by borrowing techniques from the first two chapters:

- If the coefficients are constant and the right hand side is zero, we use the same methods as in Chapter 2.
- If the coefficients are constant and the right hand side is of a particular form, we can use the method of undetermined coefficients.
- If we know one solution, we can reduce our order n equation into an order $n-1$ equation using the method of reduction of order.

Furthermore, we still know the form of a general solution of a non-homogeneous linear equation as a sum of a complementary and a particular solution; and the Wronskian can still be used to check for linear independence of solutions. The following theorem tells us that the solution space for a homogeneous equation is exactly dimension n.

Theorem

Suppose we have an equation of the form:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_0 y = 0$$

Where a_i are continuous on an interval. Then, if we find $y_1, y_2, ...y_n$ linearly independent solutions to the equation, then every solution can be expressed as a linear combination of $y_1, ..., y_n$.

We call such a solution set a **fundamental set of solutions**.

0.2 Variation of Parameters

Finally, we arrive at the most important tool we have for solving a non-homogeneous nth order linear differential equation.

Suppose we have an equation of the form:

$$y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_0 y = f(t)$$

In other words, the first coefficient is 1.

Taking the associated homogeneous equation, let $y_1, ..., y_n$ be its fundamental solutions. Then we try to find a solution to the non-homogeneous equation of the form:

$$y = u_1 y_1 + u_2 y_2 + ... + u_n y_n$$

Where u_i are functions.

To solve uniquely for each u_i, we need n equations. To find them, we first take the derivative of y:

$$y' = \sum_{i=1}^{n} u_i' y_i + u_i y_i'$$

To make things easier for us, we set the condition $\sum_{i=1}^{n} u_i' y_i = 0$. Taking the second derivative, we get:

$$y' = \sum_{i=1}^{n-1} u_i' y_i + u_i y_i'$$

And similarly, we set the condition $\sum_{i=1}^{n} u_i' y_i' = 0$. Repeating this process until we hit the $n-1$th derivative, we get $n-1$ conditions. The final condition comes from substituting in our trial solution into the differential equation:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = f(t)$$

Now substituting in $y^{(n)} = \sum_{i=1}^{n} u_i y_i^{(n)} + u_i' y_i^{(n-1)}$, we get:

$$\sum_{i=1}^{n} u_i y_i^{(n)} + u_i' y_i^{(n-1)} + a_{n-1} \sum u_i y_i^{(n-1)} + ... + a_0 \sum u_i y_i = f(t)$$

But note that for each y_i, since it is a solution to the associated homogeneous equation, we have:

$$y_i^{(n)} + a_{n-1} y_i^{(n-1)} + ... + a_0 y_i = 0$$

So most of these terms drop out! The only terms that remain give us:

$$\sum_{i=1}^{n} u_i' y_i^{(n-1)} = f(t)$$

To summarize, our conditions look like:

$$\begin{aligned} u_1' y_1 + ... + u_1' y_n &= 0 \\ u_1' y_1' + ... + u_1' y_n' &= 0 \\ &\vdots \\ u_1' y_1^{(n-1)} + ... + u_1' y_n^{(n-1)} &= f(t) \end{aligned}$$

And to put this in matrix form:

$$\begin{bmatrix} f_1 & \cdots & f_n \\ \vdots & \ddots & \vdots \\ f_1^{n-1} & \cdots & f_n^{n-1} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}' = \begin{bmatrix} 0 \\ \vdots \\ f(t) \end{bmatrix}$$

So, as long as $W \neq 0$, we can solve for u_i' and thus integrate to find u_i.

Summary of Method Suppose we have an equation of the form:

$$y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_0 y = f(t)$$

Where we have a set of fundamental solutions y_i to the associated homogeneous equation. Then, we can write a solution as:

$$y = u_1 y_1 + ... + u_n y_n$$

Where u_i are found by solving the equation:

$$\begin{bmatrix} f_1 & \cdots & f_n \\ \vdots & \ddots & \vdots \\ f_1^{n-1} & \cdots & f_n^{n-1} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}' = \begin{bmatrix} 0 \\ \vdots \\ f(t) \end{bmatrix}$$