Differential Geometry, Part VII: Riemmanian Geometry

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We now generalize the concepts built up in the previous discussion of geometry on surface in $\mathbb{R}^3$. The most important element of our earlier discussion was the Euclidean dot product. The simple generalization, then, is to define a general inner product on a vector space (namely, the tangent space). From the inner product alone we can construct a rich intrinsic geometry on surfaces which may be quite different from the Euclidean case.

0.1 Geometric Surfaces

Definition: An **inner product** on a vector space V is a real-valued, bilinear, symmetric, and positive definite (non-negative for all inputs) function $\langle v, w\rangle$ on pairs of vectors.

Now that we have something like a dot product, we can define the generalization of a surface.

Definition: A geometric surface is an abstract surface with an inner product on each tangent space.

In particular this means that if V, W are differentiable vector fields on a surface M, then $\langle V, W\rangle$ is a differentiable scalar function on M.

Definition: A metric tensor is a function on all pairs of vectors such that:

$$g_p(v, w) = \langle v, w\rangle_p$$

A metric tensor is thus similar to a 2-form, but instead of being skew-symmetric, it is symmetric. In its arguments. Loosely, we have:

$$\text{surface} + \text{metric tensor} = \text{geometric surface}$$

Now, we define a **frame field** on a surface M, as before, as a pair of unit vector fields E_1, E_2 so that:

$$\langle E_i, E_j\rangle = \delta_{ij}$$

And from this we have the dual 1-forms:

$$\theta_i(E_j) = \delta_{ij}$$

Finally, we carry over the connection form ω_{12}, which uniquely satisfied the structural equations:

$$\begin{aligned} d\theta_1 &= \omega_{12} \wedge \theta_2 \\ d\theta_2 &= \omega_{21} \wedge \theta_1 \end{aligned}$$

In a neighborhood p of M, we get a differentiable angle function θ so that for any unit tangent vector field $\bar{E}_1$:

$$\bar{E}_1 = \cos\varphi E_1 + \sin\varphi E_2$$

So indeed we can pick:

$$\bar{E_2} = -\sin\varphi E_1 + \cos\varphi E_2$$

And we get another (positively oriented) frame field. We can indeed pick $-\bar{E_2}$ to get a negatively oriented frame.

Lemma Let E_1, E_2 and $\bar{E_1}, \bar{E_2}$ be frame fields on some region in M:

$$\bar{\omega_{12}} = \pm(\omega_{12} + d\varphi)$$
$$\bar{\theta_1} \wedge \bar{\theta_2} = \pm(\theta_1 \wedge \theta_2)$$

Where we pick the sign depending on whether the frames have the same or opposite orientations. This proof follows by simple algebra to write θ_1, θ_2 in terms of the barred versions; then, take exterior derivatives and apply the first structural equations. By the uniqueness of solutions to the structural equations, we get our first result; by applying both wedge products to (E_1, E_2), we get the second result.

Definition: A manifold with a metric tensor is called a **Riemannian manifold**.

So indeed, a geometric surface is simply a 2-dimensional Riemmanian manifold.

0.2 Gaussian Curvature

Though we initially defined the Gaussian curvature in terms of the shape operator, the Theorema Egregium showed us we need not consider the ambient context; instead, we can calculate K from purely properties of the surface which are intrinsic to the surface itself. So we claim:

Theorem On a geometric surface M, there is a unique real-valued function K so that for every frame field on M, the second structural equation:

$$d\omega_{12} = -K\theta_1 \wedge \theta_2$$

The proof follows from the basis formulas, which tell us that there is a unique K satisfying the above (the second structural equation). But why does this hold for all frame fields? Say, that for example:

$$d\bar{\omega_{12}} = -\bar{K}\bar{\theta_1} \wedge \bar{\theta_2}$$

We wish to show that $K = \bar{K}$ on the overlap of the domains of the two frame fields. By our earlier theorem, we have:

$$\bar{\omega_{12}} = \omega_{12} \pm d\varphi$$

Where the sign is determiend depending on orientation. Taking the exterior derivative of both sides and noting that $d^2 = 0$, we indeed get that:

$$d\bar{\omega_{12}} = d\omega_{12}$$
$$\bar{K}\bar{\theta_1} \wedge \bar{\theta_2} = K\theta_1 \wedge \theta_2$$

Since both of the wedge products are nonzero, indeed $\bar{K} = K$. So indeed, the first structural equations uniquely determine the connection form ω_{12}, and the second structural equations uniquely determine K everywhere.

Example Let T be a torus of revolution with the usual parametrization:

$$x(u, v) = ([R + r\cos u]\cos v, [R + r\cos u]\sin v, r\sin u)$$

We define the metric with:

$$\langle x_u, x_u \rangle = 1$$
$$\langle x_u, x_v \rangle = 0$$
$$\langle x_v, x_v \rangle = 1$$

This determines a unique inner product at each point and consequently at each tangent plane. But this torus is flat! Recall:

$$\langle v, w \rangle_M = \langle F_\star(v), F_\star(w) \rangle_N$$

For any isometry F between surfaces M, N. And indeed x is a regular mapping, so we can indeed push forward an inner product on $\mathbb{R}^2$ to one on M:

$$\langle U_1, U_2 \rangle_{\mathbb{R}^2} = \langle x_\star(U_1), x_\star(U_2) \rangle_M$$

And this is indeed the inner product we just defined, since $x_\star(U_1) = x_u$. What we have done is pull back this torus under x to get the plane. By Gauss' Theorema Egregium, K is preserved, so we have defined a flat torus. However, this torus is compact, and we know from earlier discussion that in $\mathbb{R}^3$ that a compact surface has positive curvature. Therefore, the flat torus does not exist in $\mathbb{R}^3$.

Corollary For the plane $\mathbb{R}^2$ with the metric tensor $\langle v, w \rangle = \frac{v \cdot w}{h^2(p)}$ for some function $h(u, v) > 0$, the Gaussian curvature is:

$$K = h(h_{uu} + h_{vv}) - (h_u^2 + h_v^2)$$

The proof is simple. The identity map $\mathbb{R}^2 \to M$ is a patch with $E = G = \frac{1}{h^2}, F = 0$. Recall earlier that we proved we can compute K purely in terms of E, F, G; using this formula the result follows by algebra.

Example For example, we can define the so-called hyperbolic plane with $h(u,v) = 1 - \frac{u^2+v^2}{4}$ defined in the disk $u^2 + v^2 < 4$. By our earlier lemma, $K = h(h_{uu} + h_{vv}) - (h_u^2 + h_v^2) = -1$.

Note that any regular mapping $F : M \to N$ pulls back a metric tensor on N to one on M, but in general you can only push forward a metric via a diffeomorphism (which has a smooth inverse). The essential problem is that if $F(p_1) = F(p_2)$, then inner products might transfer differently at those two points. We address this issue with the following proposition.

Proposition Suppose F is a regular mapping from a geometric surface M to a surface N. Suppose when $F(p_1) = F(p_2)$, that there is an isometry G_{12} from a neighborhood of p_1 to a neighborhood of p_2 such that:

$$
\begin{aligned}
FG_{12} &= F \\
G_{12}(p_1) &= p_2
\end{aligned}
$$

Then, there exists a unique metric tensor on N which makes F a local isometry.

So indeed, this gives us a sufficient condition for the pushforward of a metric under a regular mapping to be well-defined.

Proof Since F is a local isometry, the inner product is completely determined on N by the isometry. F is regular, so for every p_1 in M with $F(p_1) = q$, there are unique vectors v_1, w_1 so that:

$$
\begin{aligned}
F_\star(v_1) &= v \\
F_\star(w_1) &= w
\end{aligned}
$$

And we write in this case that $\langle v, w\rangle_N = \langle v_1, w_1\rangle_M$. What if $F(p_2) = q$ and $p_1 \neq p_2$? The same argument holds, and we have a unique pair v_2, w_2 at p_2 which are pushed forward by the tangent map of the isometry to v, w. We want to show:

$$\langle v_1, w_1\rangle_M = \langle v_2, w_2\rangle_M$$

Let $G = G_{12}$ be an isometry. Then, $FG = F$, so by the chain rule, $F_\star G_\star = F_\star$. Therefore, $G_\star$ carries v_1, w_1 to v_2, w_2, and since G is an isometry it indeed preserves the inner product.

0.3 The Covariant Derivative

We want to generalize the concept of the covariant derivative, so that: - $\nabla_V W$ at p is the rate of change of W in the direction of $V(p)$. - $\omega_{12}(V) = \langle \nabla_V E_1, E_2\rangle$.

Lemma Assume there is a covariant derivative ∇ defined on M with linearity and the Leibniz property, satisfying the connection form equation as well. Then, ∇ satisfies the **connection equations**:

$$
\begin{aligned}
\nabla_V E_1 &= \omega_{12}(V) E_2 \\
\nabla_V E_2 &= \omega_{21}(V) E_1
\end{aligned}
$$

And for a vector field $W = f_1E_1 + f_2E_2$:

$$\nabla_V W = (V[f_1] + f_2\omega_{21}(V))E_1 + (V[f_2] + f_1\omega_{12}(V))E_2$$

This is called the **covariant derivative formula**.

Proof Let $\nabla_V = \langle \nabla_V E_1, E_1\rangle E_1 + \langle \nabla_V E_1, E_2\rangle E_2$, by orthonormal expansion. Then, we have:

$$0 = V[\langle E_1, E_1\rangle] = 2\langle \nabla_V E_1, E_1\rangle$$

So indeed, $\nabla_V E_1 = \omega_{12}(V)E_2$, since the covariant derivative has no component in the direction of E_1. The covariant derivative formula then follows by siply applying the chain rule to $W = f_1E_1 + f_2E_2$.

Theorem This theorem is often called the fundamental theorem of Riemannian Geometry. On each surface M, there exists a unique covariant derivative ∇ satisfying: - The linearity and Leibniz properties. - $\omega_{12}(V) = \langle \nabla_V E_1, E_2\rangle$. For every frame field E_1, E_2.

The surprising upshot of this theorem is that there is a single covariant derivative which satisfies the connection form condition for *all* frame fields. The preceding lemma shows that for each frame, there is at most one covariant derivative (the one specified by the covariant derivative formula above). So we wish to prove that there is at least one covariant derivative.

Proof First, we consider the local definition. For a frame field E_1, E_2 on a region in M, we know how to define the covariant derivative by the covariant derivative formula; we can check that the product rule is satisfied and linearity holds usig that formula. Then, to prove the condition regarding the connection forms, set $W = E_1$.

Secondly, we deal with the issue of consistency. Do our local definitions agree for two different frame fields? We only need to check that two frame fields defined on the same region yields covariant derivatives which agree on $W = E_1, W = E_2$. Assume WLOG that the two frames have the same orientation. Then:

$$\begin{aligned}\bar{E_1} &= \cos\varphi E_1 + \sin\varphi E_2 \\ \nabla_V \bar{E_1} &= \sin\varphi(-V[\varphi] + \omega_{21}(V))E_1 + \cos\varphi(V[\varphi] + \omega_{12}(V))E_2\end{aligned}$$

But we have $\bar{\omega_{12}} = \omega_{12} + d\varphi$, and so we our above equation reduces to:

$$\begin{aligned}\nabla_V \bar{E_1} &= \bar{\omega_{12}}(V)(-\sin\varphi E_1 + \cos\varphi E_2) \\ &= \bar{\omega_{12}}(V)\bar{E_2} \\ &= \bar{\nabla}_V \bar{E_1}\end{aligned}$$

And similarly for E_2.

In particular, for $\mathbb{R}^2$, we get the covariant derivative defined earlier: if $W = \sum f_i U_i$, then $\nabla_V W = \sum V[f_i]U_i$.

For curves, we want to define something like $\nabla_{\alpha'} Y$ as we did before. But the quantities involved are not necessarily defined on open sets. So we simply decide:

$$Y' = (f_1' + f_2\omega_{21}(\alpha'))E_1 + Y' = (f_2' + f_1\omega_{12}(\alpha'))E_2$$

What we have done is simply adapt the covariant derivative by replacing $V[f_i]$ with f_i'.

Definition: A vector field V on α in a geometric surface is parallel if $V' = 0$ along α.

In other words, V is paralle if the covariant derivative vanishes. Note that in this case, $\langle V, V\rangle' = 2\langle V', V\rangle = 0$, so a parallel vector field has perfect length.

LEmma If $alpha$ is a curve, and v a tangent vector at $p = \alpha(t_0)$, then there is a unique parallel vector field V so that $V(t_0) = v$.

This follows from the uniqueness and existence theorems for differential equations. IN this case, the conditions are: $V'(t) = 0$ and $V(t_0) = v$. It is not hard to see by the previous lemma that the angle function has the corresponding conditions $\varphi' + \omega_{12}(\alpha') = 0$, and $\varphi(t_0)$ is the angle between $E_1(p)$ and v. This allows us to get an explicit formula for φ by integrating. This is called the parallel translation of v at p along α.

If v is taken by parallel translation around a closed curve, it might not end up as v when it returns to the same spot, a phenomenon called holonomy. We know, however, that in particular:

$$\varphi(b) - \varphi(a) = -\int_\alpha \omega_{12}$$

So we call this quantity ψ_α, the holonomy angle of α.

What is the relation between our new concept of the covariant derivative and the one we earlier defined in Euclidean space? This is illuminated in the following lemma.

Lemma If V, W are tangent vector fields to M, and if $\tilde{\nabla}$ denotes the Euclidean covariant derivative, then: - ∇_V, W is the component of $\tilde{\nabla}_V W$ which is tangent to M. - $\tilde{\nabla}_V W = \nabla_V W + (S(V) \cdot W)U$ Where $S(V)$ is the shape operator and U the associated unit normal.

To prove this, we need only prove the second condition, which implies the first. But we have by definition using the connection forms:

$$\begin{aligned}\tilde{\nabla}_V E_1 &= \sum \omega_{ij}(V)E_j \\ &= \omega_{12}(V)E_2 + \omega_{13}(V)E_3\end{aligned}$$

And the definition of the covariant derivative on surfaces gives:

$$\nabla_V E_1 = \omega_{12}(V)E_2$$

And, since $E_3 = U$, we have:

$$\begin{aligned}\omega_{13}(V)E_3 &= (\tilde{\nabla}_V E_1 \cdot E_3)E_3 \\ &= (\tilde{\nabla}_V E_3 \cdot E_1)E_3 \\ &= (S(V) \cdot E_1)U\end{aligned}$$

We can do the same thing and instead take the covariant derivative of E_2 to get out result, and then prove the result for an arbitrary $W = f_1 E_1 + f_2 E_2$ by calculus.

0.4 Geodesics

We now can define geodesics without using the unit normal.

Definition: A curve γ in a geometric surface is a geodesic if $\gamma'' = 0$; equivalently, γ' is parallel; equivlaently, γ' has constant length.

Since acceleration is preserved by isometries, geodesics are isometric invariants under our new definition. Indeed, that means if γ is a geodesic, then $F \circ \gamma$ is also a geodesic. We use the notation:

$$\begin{aligned}\alpha' &= v_1 E_1 + v_2 E_2 \\ \alpha'' &= A_1 E_1 + A_2 E_2\end{aligned}$$

By definition, α is a geodesic iff $A_1 = A_2 = 0$. In particular, if we apply the above definition for the covariant derivative of α, we arrive at the conditions:

$$\begin{aligned}A_1 &= v_1' + v_2 \omega_{21}(\alpha') = 0 \\ A_2 &= v_2' + v_1 \omega_{12}(\alpha') = 0\end{aligned}$$

We can indeed use an associated frame field, which gives the following equivalent conditions (I will not go through the proof here):

$$\begin{aligned}A_1 &= a_1'' + \frac{1}{2E}(E_u a_1'^2 + 2E_v a_1' a_2' - G_u a_2'^2) = 0 \\ A_2 &= a_2'' + \frac{1}{2G}(-E_v a_1'^2 + 2G_u a_1' a_2' + G_v a_2'^2) = 0\end{aligned}$$

We now arrive at a theorem which gives us the ability to construct geodesics easily.

Theorem Given a tangent vector v to M, there exists a unique geodesic γ, defined around 0, so that $\gamma(0) = p, \gamma'(0) = v$.

To prove this, let x be an orthogonal patch around $p = x(u_0, v_0)$ (orthogonal meaning $F = 0$). Let $v = cx_u + dx_v$. Then our above conditions become:

$$\begin{aligned}a_1'' &= f_1(a_1, a_2, a_1', a_2') \\ a_2'' &= f_2(a_1, a_2, a_1', a_2')\end{aligned}$$

And with the initial conditions $(a_1, a_2)(0) = (u_0, v_0)$ and $(a_1', a_2')(0) = (c, d)$. By the existence and uniqueness of differential equations, we get our result.

Note that this allows us to extend geodesics infinitely. Let us fix a tangent vector v. Then at every point, we have a unique geodesic γ pointing in the direction of v defined in a neighborhood of 0. If we pick a point near the edge of that neighborhood, we define another unique geodesic, and so on. By gluing together all these geodesics, we get what's called a **maximal geodesic**.

Definition: A geometric surface M is complete if every maximal geodesic γ_v in M is defined on the whole real line $\mathbb{R}$.

Here, we only need to parametrize the maximal geodesics by their direction v. In a complete geometric surface, a maximal geodesic runs infinitely. For example, on a sphere, a great circle is a geodesic which repeats itself periodically and thus runs forever.

As it turns out, all compact geometric surfaces are complete.

Lemma Let E_1, E_2 be a frame field with constant speec so that α' is never orthogonal to E_2. If $A_1 = 0$, then α is a geodesic.

The proof is straightforward. Since α' is constant length, we have $\langle \alpha', \alpha' \rangle' = 2\langle \alpha'', \alpha' \rangle = 0$. So indeed:

$$\langle A_1 E_1 + A_2 E_2, \alpha' \rangle = A_1 \langle E_1, \alpha' \rangle + A_2 \langle E_2, \alpha' \rangle = 0$$

By hypothesis, $A_1 = 0$. Since E_2 is not orthogonal to α', we must have $A_2 = 0$, and thus indeed α is a geodesic.

In particular, if you pick E_1, E_2 to be frame fields so that $\langle x_u, x_v \rangle = \delta_{ij}$, then our above conditions on A_1, A_2 can be simplified. Adding and integrating, we get:

$$E a_1'^2 + G a_2'^2 = \text{const}$$

We now introduce a critical notion for the next section. If $\beta : I \to M$ is aunit-speed curve on an oriented surface, then we can define $T = \beta', N = J(T)$ to get an analogue of the Frenet frame field. We then get an analogue for curvature, the **geodesic curvature** k_g:

$$T' = k_g N$$

As a direct corollary, if φ is the angle between E_1 and β', then we have:

$$k_g = \frac{d\varphi}{ds} + \omega_{12}(\beta')$$

Note that in the case of Euclidean space, this confirms our notion that curvature measures the ``rate of turning'' of the tangent vector.

We get this by setting $Y = T$, $\alpha = \beta$, and applying the definition of the covariant derivative for vector fields with respect to curves on geometric surfaces. Another interesting consequence of the definition of geometric curvature is that:

$$\alpha' = vT$$
$$\alpha'' = \frac{dv}{dt}T + k_g v^2 N$$

Where v is the speed of the curve α. This formual gives is an easy condition to see if a curve is a geodesic in terms of its geodesic curvature:

Lemma A regular curve α is a geodesic iff it has constant speed and $k_g = 0$.

Since the speed $v > 0$ by regularity, by the above formula $\alpha''0$ is equivalent to the condition that $\frac{dv}{dt} = k_g = 0$. This means that α'' is collinear to the tangent vector, as in our earlier definition of geodesics (curves whose accelerations were normal to the surface). We can easily turn any curve with $k_g = 0$ (called a **pre-geodesic**) into a geodesic, then, by reparametrizing and setting speed to be constant.

0.5 The Gauss-Bonnet Theorem

Definition: $\alpha : [a, b] \to M$ is a regular curve segment in an oriented geometric surface. We define the **total geodesic curvature** to be the quantity:

$$\int_\alpha k_g ds = \int_{s(a)}^{s(b)} k_g(s(t)) \frac{ds}{dt} dt$$

The total geodesic curvature is the analogue of the total Gaussian curvature, and we will soon see there is a link between them.

Lemma Let α be a regular curve in M which has an oriented frame field E_1, E_2. Then we have:

$$\int_\alpha k_g ds = \varphi(b) - \varphi(a) + \int_\alpha \omega_{12}$$

Where φ is the angle from E_1 to α' along α. This lemma follows from simply integrating our earlier formula for the geodesic curvature.

We now begin the road to proving the famous Gauss-Bonnet theoerem, which tells us that the total curvature is intricately connected to topological invariants of a surface.

Definition: Let $x : R \to M$ be a one-to-one regular patch from a 2-segment. We define the exterior angle ϵ_j of x at the vertice p_j to be the angle between the relevant tangent vectors of the boundary curves. The interior angle i_j is similarly $\pi - \epsilon_j$; the interior and exterior angles add up to π.

Theorem Let $x : R \to M$ be a one-to-one, regular mapping to a 2-segment in a geometric surfcae M. If dM is an area form, then:

$$\int\int_x K dM + \int_{\partial x} k_g ds + \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 = 2\pi$$

This is often called the **Gauss-Bonnet** formula.

Proof We pick the associated frame field $E_1 = \frac{x_u}{\sqrt{E}}, E_2 = J(E_1)$, where J is the rotation operator. Then we have $dM(E_1, E_2) = 1$; we picked the positive orientation. By the second structural equation:

$$\begin{aligned} d\omega_{12} &= -K\theta_1 \wedge \theta_2 \\ &= -KdM \end{aligned}$$

Now, by Stokes Theorem, we have:

$$\int_{\partial x} \omega_{12} = \int \int_x d\omega_{12} = \int \int_x -KdM$$
$$\int_{\partial x} \omega_{12} + \int \int_x KdM = 0$$

But recall earlier we defined:

$$\int_\alpha k_g ds = \varphi(b) - \varphi(a) + \int_\alpha \omega_{12}$$

Adding up the integrals for each of the four boundary curves, we get the required result.

Definition: A regular decomposition D of a surface M is a finite collection x_i of rectangles which intersect at a common vertex or edge.

This is essentiall what we did earlier with the concept of a paving, but slightly modified.

Theorem Every compact surface M has a rectangular decomposition. We won't prove this here, as it is a result from topology.

Theorem In dimension 2, two surfaces are diffeomorphic iff they are homemomorphic. We won't prove this here.

What the above theorem implies is that topological invariants (those preserved by homeomorphisms) are now in fact preserved by diffeomorphisms. This reduces our work considerably, since diffeomorphisms give us differentiability.

Theorem Let D be a rectangular decomposition of a compact surface. let v, e, f denote the number of vertices, edges, faces in D. Then $v - e + f$, called the **Euler characteristic** $\chi(M)$, is the same for each rectangular decomposition.

This theorem from topology tells us that the Euler characteristic does not depend on our rectangularization. So, if we have two separate rectangular decompositions, we can construct a diffeomorphism by varying smoothly between them, and thus we know that diffeomorphic surfaces have the same Euler characteristic.

Theorem If M is a comapct, connected, orientable surface, there is a unique positive integer h such that M is diffeomorphic to $\Sigma[h]$, which is the sphere with h handles added on.

A handle can be constructed as follows. Take any rectangular decomposition of M, and glue together one ``tile'' from M to a tile from a rectangular decomposition of a torus. By looking at the case of the sphere which has Euler characteristic 2, when we add on a handle we get a surface diffeomorphic to a torus, which has Euler characteristic 0. In general, adding h handles reduces the Euler characteristic of the sphere by $2h$.

As a direct corollary, compact orientable surfaces M, N have the same Euler characteristic if and only if they are diffeomorphic.

To prove this statement, consider that $\chi(M) = \chi(N)$ means that both surfaces are diffeomorphic to a sphere with h handles; so they are diffeomorphic to each other.

We can now state the incredible Gauss-Bonnet Theorem.

Theorem (Gauss-Bonnet) For a compact, orientable surface, we have:

$$\int\int_M KdM = 2\pi\chi(M)$$

This theorem connects a topological invariant (the Euler characteristic) to a geometric/isometric invariant (the Gaussian curvature). In fact, this says that the Gaussian curvature is a topological invariant.

Proof I will sketch the proof here, avoiding some details. First, we orient M by an area form dM, and construct a rectangular decomposition D which is positively oriented (we can simply reorder the tangent vectors until we get positive orientations on each rectangle). Then, D is an oriented paving. We can now integrate to obtain:

$$\int\int_M KdM = \sum\int\int_{x_i} KdM$$

But by the ealrier Gauss-Bonnet formula, we have:

$$\sum\int\int_{x_i} KdM = -\int_{\partial x_i} k_g ds - 2\pi + i_1 + i_2 + i_3 + i_4$$

But on the intersection, we note that since all the rectangles have the same orientaton, the Gaussian curvatures on boundary curves cancel out! This is similar to a common trick you see in integration. If we split up a rectangle into four smaller rectangles and integrate over each, the only terms that do not cancel are associated with the boundary curves. So we are left with:

$$\sum\int\int_{x_i} KdM = \sum -2\pi + i_1 + i_2 + i_3 + i_4$$

Where we are summing over the number of tiles in our paving (the number of faces). So we can rewrite the sum above as:

$$\sum -2\pi + i_1 + i_2 + i_3 + i_4 = -2\pi f + \sum i_1 + i_2 + i_3 + i_4$$

Where f is the number of faces. Note that around every vertex, we have four interior angles which add up to 2π. Therefore, we can reduce the sum on the right to obtain:

$$\sum \int \int_{x_i} KdM = 2\pi(v - f)$$

And finally, we try to count the number of edges in a surface. If we simply multiply the amount of faces by four, we are double counting, since every edge appears on two faces. So we have $4f = 2e$. Equivalently above:

$$\sum \int \int_{x_i} KdM = 2\pi\chi(M)$$

And we are done.

0.6 Applications of Gauss-Bonnet

We know run through some simple consequences of the Gauss-Bonnet Theorem.

First, there is no metric on a sphere $\sum$ with $K < 0$. If there was, then the total Gaussian curvature would be negative. But by Gauss-Bonnet, the total Gaussian curvature is $2\pi\chi(M) = 4\pi$, which is positive.

Second, every compact, orientable geometric surface M which has $K > 0$ is diffeomorphic to a sphere. If the geometric surface has positive total Gaussian curvature, then $\chi(M)$ is positive by Gauss-Bonnet. But if we add even one handle to a sphere, the Euler characteristic drops to 0. Therefore, any geometric surface is diffeomorphic to a sphere.

Theorem The following are equivalent on a compact, orientable surface. - There exists a non-vanishing tangent vector field on M. - The Euler characteristic of M is 0. - M is diffeomorphic to a torus.

First, we prove that the first statement implies the second. We construct an associated frame field from the non-vanishing vector field:

$$E_1 = \frac{V}{|V|}$$
$$E_2 = J(E_1)$$

By the second structural equation, $d\omega_{12} = -KdM$.

By Gauss-Bonnet:

$$\int \int_M KdM = 2\pi\chi(M)$$

By Stokes' Theorem:

$$\int\int_M KdM = -\int\int_M d\omega_{12}$$

But for the associated frame field, $\omega_{12} = 0$, so indeed this means the integral is 0 and thus so is the Euler characteristic.

Next, we prove the second statement implies the third. If the Euler characteristic is 0, then M is diffeomorphic to a sphere with one handle, which is a torus.

Finally, we prove the third statement implies the first. We use x_u, x_v from the usual parametrization of the torus; either is a non-vanishing tangent vector field.

0.7 Summary

Much of our discussion of surfaces revolved around the Euclidean dot product. However, in this chapter we showed that we can create arbitrary metric spaces on surfaces to obtain a Riemannian manifold. On a Riemannian manifold, many concepts and ideas see analogous results: differential forms, frame fields, the structural equations, covariant derivatives, curvature, and geodesics. An interesting result we can prove is that the total curvature is indeed a topological quantity; thus, we have established a fundamental link between geometry and topology.