Differential Equations, Part I: First Order Differential Equations

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This post will be a brief overview of first-order differential equations. I won't be focusing mainly on proofs here, but instead techniques for solving equations, as well as existence and uniqueness theorems. Uniqueness tells us that there is only one solution to the differential equation in the specified interval.

These notes require cursory knowledge of linear algebra and multivariable calculus.

0.1 Integrating Factors

Integrating factors can be used to solve first order equations of degree 1, in other words equations of the form:

$$y' + P(t)y = Q(t)$$

The idea is to multiply through by a factor $\mu(t)$ suitably chosen so that the left hand side is an instance of the product rule. This integrating factor turns out to be:

$$\mu(t) = e^{\int P(t)dt}$$

Multiplying through, we are left with the product rule on the left hand side, yielding a new simpler differential equation:

$$(\mu(t)y(t))' = \mu(t)Q(t)$$

And this shouldn't be difficult to integrate in general.

0.2 Separable Equations

A separable differential equation can be rewritten in the form:

$$N(y)\frac{dy}{dx} = M(x)$$

Rewriting:

$$N(y)dy = M(x)dx$$

Integrating both sides gives a solution for y.

0.3 Exact Equations

We start with a differential equation of the form:

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$$

Or equivalently, we can write:

$$M + N\frac{dy}{dx} = 0$$

Now, we can arrive at this same differential equation in another way. Suppose we have an implicit function $f(x,y) = c$, which describes some sort of level set of a curve. For example, we could have circles, with $f(x,y) = x^2 + y^2 = c$ defining a circle of radius $\sqrt{c}$.

Then, by the Implicit Function Theorem, assuming the Jacobian matrix of f is invertible, then we can write:

$$f(x, y(x)) = c$$

In other words, we can write y in terms of x. Furthermore, dy/dx, the rate of change, can be computed:

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

So instead of studying the above differential equation, we study the related implicitly defined surface with $f(x,y) = c$ for some constant, such that $f_x = M, f_y = N$ (where f_x denotes the partial derivative $\frac{\partial f}{\partial x}$). It isn't too hard to find such a function f in general by integrating M with respect to x and N with respect to y.

We could equivalently say that we can check for exactness with the necessary and sufficient condition that $M_y = N_x$, and then use the above method.

Example Let us look at: $\frac{dy}{dx} = -\frac{x}{y}$ defined in an open set away from $y = 0$. We have $M = x, N = y$ (note: you can pick other pairs, such as $M = 2x, N = 2y$ and the only thing that will change is our choice of constant). It is not hard to see by integration that we can pick $f(x,y) = x^2 + y^2 = c$. So instead of solving the above differential equation, we simply use the initial conditions to find c and then solve for y in terms of x.

0.4 Existence and Uniqueness

How do we assure existence and uniqueness of solutions to a first order differential equation?

Theorem

If $p(x), g(x)$ are continuous in an open interval I containing $x = x_0$, then there exists a unique function $y(x)$ satisfying the equation:

$$y' + p(x)y = g(x)$$

As well as the initial condition $y(x_0) = 0$ for each $x \in I$.

The proof of existence is given by construction in the ``integrating factors'' method. Uniqueness is guaranteed by the continuity of p, g. This tells us that linear first order differential equations have unique solutions.

Theorem

Let $f(x, y)$ be a real-valued function with continuous derivative $\frac{\partial f}{\partial y}$ which is continuous in some rectangle in the plane containing the point (x_0, y_0). As a stronger condition, we could simply set f to be differentiable in some rectangle. Then in some interval $(x_0 - h, x_0 + h)$, there is a unique function $y = \phi(x)$ such that:

$$\phi(x)' = f(x, y)$$

$$\phi(x_0) = y_0$$

For a given y_0.

This tells us that, given certain conditions, we can even have unique solutions to non-linear first order differential equations satisfying initial conditions (namely ,we just need $y' = f(x, y)$ continuous with f_y continuous).

Euler's Method To approximate a solution, we could set a sufficiently small parameter h and walk a distance h the tangent line at any point. The error is the quadratic term of the Taylor expansion; by making h sufficiently small, we can remain close to the curve. We can also create n equations, called ``finite difference equations'', expressing y_n, a value at stage n, in terms of y_{n-1} recursively. In these cases, it is useful to consider the equilibrium solutions $y_n = y_{n+1}$.